Partially Reversible Capital Investment under Demand Ambiguity*

Motoh Tsujimura†
Faculty of Commerce, Doshisha University

Abstract

This paper investigates a firm's partially reversible capital investment problem when the output demand is ambiguous. We adopt the Choque-Brownian motion process to incorporate the ambiguity of demand. To solve the firm's problem, we formulate it as a singular stochastic control problem. Then, we use variational inequalities and derive the optimal investment strategy which is described by two thresholds to invest in and reduce the capital. Furthermore, we obtain useful insights for the firm's investment decision-making through the comparative static analysis. We find that a higher volatility and ambiguity aversion discourage the capital investment.

Keywords: Capital investment; Ambiguity; Choque-Brownian motion; Singular Stochastic Control; Variational inequalities

1 Introduction

Capital investments have been a central issue in a firm's decision making for decades, as it has an impact on firm value. The investment decision-making is influenced by future economic conditions. Then, it is important for the firm's manager to manage the effects of uncertainty caused by future economic conditions. In particular, output demand/price uncertainty is a major uncertainty and its effects have been investigated by many researchers. As representative studies, Hartman (1972), Able (1983), and Abel and Eberly (1994) investigate the effects of output price uncertainty on a firm's capital investment. They formulated a competitive firm's capital investment problem, which is to maximize the expected present value of operating profit less total investment costs, that is, the maximizing the firm's value by choosing capital investment rate at each time. They showed that the output price uncertainty promotes capital investment. See also Caballero (1991) for a discussion of capital investment under uncertainty.

Another important factor of capital investment is irreversibility, which comes from the sunk cost associated with investment. It is natural that the irreversibility discourages capital investment. Real option analysis reveals that uncertainty postpones investment timing when the

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†Corresponding author
Address: Karasuma-Higashi-iru, Imadegawa-dori, Kamigyo-ku, Kyoto, 602-8580 JAPAN
Phone: +81-75-251-4582; E-mail: mtsujimu@mail.doshisha.ac.jp

investment is irreversible. Because of the irreversibility, a firm then becomes more cautious about investing in capital under uncertainty. See Dixit and Pindyck (1994) for more details of real option analysis.

The concept of uncertainty has been received attention from researchers. One of most famous concept is provided by Knight (1921). He defined two kinds of uncertainty: risk and uncertainty. The probability of an outcome is not uniquely determined under the uncertainty, while under the risk it is. The latter is termed Knightian uncertainty or ambiguity. In this paper, we use the term ambiguity. See, for example, Camerer and Weber (1992), Etner et. al. (2012) and Guidolin and Rinaldi (2013) for a survey of decision-making under ambiguity. Nishimura and Ozaki (2007), Trojanowska and Kort (2010), Wang (2010), and Thijssen (2011) explore the irreversible capital investment problem under ambiguity.

As another line of capital investment studies, many researchers have investigated the case in which the firm can invest in the capital as needed. See, for example, Kobila (1993), Bertola and Caballero (1994), Chiarolla and Haussmann (2005), and Motairi and Zervos (2017). They investigated the firm's capital expansion problem under uncertainty. To solve the firm's problem, they formulated it as a singular stochastic control problem. Furthermore, if there exists a secondary market of the capital or the capital can be directly sold to another firm, the investment expenditure is partially irreversible. In the partially irreversible investment, the firm's capital investment problem has been extended as the capital expansion and reduction problem. See, for example, Abel and Eberly (1996), Guo and Pham (2005), Merhi and Zervos (2007) Angelis and Ferrari (2014), Federico and Pham (2014). They also formulated the firm's problem as a singular stochastic control problem. The capital expansion problem is required one boundary for investing, while the capital expansion and reduction problem is required two boundaries for investing.

This paper explores a firm's capital investment problem when the future output demand is ambiguous. To this end, we formulate the firm's problem as a singular stochastic control problem. In this paper, the output demand ambiguity is expressed by Choque-Brownian motion, which is developed by Kast and Lapied (2010) and Kast et al. (2014). The previous studies mentioned above adopt the framework of κ-ignorance developed by Chen and Epstein (2002) in order to incorporate ambiguity. In this framework, the ambiguity affects only drift terms of associated processes. On the other hand, the framework provided by Kast et al. affects drift terms and diffusion terms of associated processes. We solve the firm's problem by using variational inequalities and derive the optimal investment strategy which is characterized by two thresholds to invest in and reduce the capital. Furthermore, we conduct a comparative static analysis on some parameters. We obtain the following two main findings: a higher volatility discourages the capital investment; if the firm's manager is more ambiguity averse, the capital investment is also postponed.

The rest of the paper is organized as follows. Section 2 describes the setup of the firm's problem. Section 3 solve the firm's problem. Next, we present the numerical analysis in Section 4. Section 5 concludes the paper.

2 The Model

Suppose that a firm produces a single output by using a capital K and sells it in competitive market. The firm's manager faces the ambiguity on the output demand X. To analyze the ambiguity, we adopt the framework developed by Kast and Lapied (2010) and Kast et al. (2014). The dynamics of X is governed by the following stochastic differential equation:

$$\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t^c, \quad X_{0^-} = x > 0, \tag{2.1}$$

where $\mu > 0$ and $\sigma > 0$ are constants. W_t^c is a generalized Wiener process with mean $m = 2c - 1$ and variance $s^2 = 4c(1-c)$:

$$\mathrm{d}W_t^c = m\mathrm{d}t + s\mathrm{d}W_t, \tag{2.2}$$

where W_t is a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$, where $\mathcal{F}_t$ is generated by W_t. c $(0 < c < 1)$ is the constant conditional Choquet capacity which indicates the firm manager's attitude toward ambiguity. Then, the output demand follows a Choquet-Brownian motion:

$$\mathrm{d}X_t = (\mu + m\sigma)X_t \mathrm{d}t + s\sigma X_t \mathrm{d}W_t, \quad X_{0^-} = x > 0. \tag{2.3}$$

If $c < \frac{1}{2}$ (resp. $c > \frac{1}{2}$), the firm manager is ambiguity averse (resp. ambiguity loving). If $c = \frac{1}{2}$, then $m = 0$ and $s = 1$. This means that $\mathrm{d}W_t^c = \mathrm{d}W_t$ and ambiguity disappears and the firm manager has perfect confidence toward the dynamics of X. The firm expands its business through accumulating the capital, while the firm reduces its business through reducing the capital. Let I_t^+ and I_t^- be the cumulative expansion and reduction of capital until t, respectively. They are right-continuous with left-hand limits adapted processes, nonnegative and nondecreasing, with $I_{0-}^+ = 0$ and $I_{0-}^- = 0$. The firm can purchase the capital at constant price $p > 0$ or sell it at constant price $(1-\lambda)p > 0$, $\lambda \in (0,1)$. Then the dynamics of the capital is given by:

$$\mathrm{d}K_t = -\delta K_t \mathrm{d}t + \mathrm{d}I_t^+ - \mathrm{d}I_t^-, \quad K_{0^-} = k\ (>0), \tag{2.4}$$

where $\delta \in (0,1)$ is a constant depreciation rate.

The firm's operating profit $\hat{\pi}$ at time t is given by:

$$\hat{\pi}(K_t, X_t) = K_t^{\alpha} X_t^{\beta}, \tag{2.5}$$

where $\alpha \in (0,1)$, $\beta > 0$. The firm's expected discounted profit $J(k, x; I^+, I^-)$ is given by:

$$\hat{J}(k, x; I^+, I^-) = \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-rt}\hat{\pi}(K_t, X_t)\mathrm{d}t - p\int_0^\infty \mathrm{e}^{-rt}\mathrm{d}I_t^+ + (1-\lambda)p\int_0^\infty \mathrm{e}^{-rt}\mathrm{d}I_t^-\right], \tag{2.6}$$

where $r > 0$ is a discount rate, $I^+ := \{I_t^+\}_{t\geq 0}$ and $I^- := \{I_t^-\}_{t\geq 0}$ and the pair (I^+, I^-) is an investment strategy. The investment strategy is admissible when $(I^+, I^-) \in \mathcal{A}$, where $\mathcal{A}$ is the set of all admissible investment strategies. In this context, it is assumed that:

$$\mathbb{E}\left[\int_0^\infty \mathrm{e}^{-rt}\hat{\pi}(K_t, X_t)\mathrm{d}t\right] < \infty, \tag{2.7}$$

and

$$\mathbb{E}\left[\int_0^\infty \mathrm{e}^{-rt}\mathrm{d}I_t^+\right] < \infty. \tag{2.8}$$

Therefore, the firm's problem is to maximize the expected discounted profit over $\mathcal{A}$:

$$\hat{V}(k, x) = \sup_{(I^+, I^-)\in\mathcal{A}} \hat{J}(k, x; I^+, I^-) = \hat{J}(k, x; I^{+*}, I^{-*}), \tag{2.9}$$

where $\hat{V}$ is the value function and (I^{+*}, I^{-*}) is the optimal investment strategy. This firm's problem (2.9) is formulated as a singular stochastic control problem.

3 Variational Inequalities

From the formulation of the firm's problem (2.9), it is able to guess that, under an optimal investment strategy, the firm's investment strategy as follows. The firm maintains the capital stock level in a region, so that whenever the capital stock is below (resp. above) the lower (resp. upper) boundary, the firm invests in (reduces) the capital. Note that the boundaries of the capital stock K are depend on the demand level X. Then, the x-k plane is divided into three region: continuation region, capital expansion region, and capital reduction region. In this paper, we change variables and replace the ratio of k and x with y for analytical tractability. Consequently, the three regions are defined by two thresholds, $\underline{y}$ and $\bar{y}$. In order to verify this conjecture of investment strategy, we solve the firm's problem (2.9) by using variational inequalities.

The variational inequalities of the firm's problem (2.9) are given as follows:

Definition 3.1 (Variational Inequalities). *The following relations are called the variational inequalities for the firm's problem (2.9):*

$$\hat{\mathcal{L}}\hat{V}(k,x) + \hat{\pi}(k,x) \leq 0, \tag{3.1}$$

$$\hat{V}_K(k,x) \leq p, \tag{3.2}$$

$$\hat{V}_K(k,x) \geq (1-\lambda)p, \tag{3.3}$$

$$[\hat{\mathcal{L}}\hat{V}(k,x) + \hat{\pi}(k,x)][\hat{V}_K(k,x) - p][(1-\lambda)p - \hat{V}(k,x)] = 0, \tag{3.4}$$

where $\hat{\mathcal{L}}$ is the operator defined by:

$$\hat{\mathcal{L}} := -\delta k \frac{\partial}{\partial K} + (\mu + m\sigma)x\frac{\partial}{\partial X} + \frac{1}{2}s^2\sigma^2 x^2 \frac{\partial^2}{\partial X^2} - r. \tag{3.5}$$

See, for example, Harrison and Taksar (1983) and Merhi and Zervos (2007) for derivation of the variational inequalities. The variational inequalities are summarized as:

$$\max\left\{\hat{\mathcal{L}}\hat{V}(k,x) + \hat{\pi}(k,x), \hat{V}_K(k,x) - p, (1-\lambda)p - \hat{V}_K(k,x)\right\} = 0. \tag{3.6}$$

Let $\hat{\mathcal{H}}$ be the continuation region given by:

$$\hat{\mathcal{H}} := \{(k,x); (1-\lambda)p < \hat{V}_K(k,x) < p, \hat{\mathcal{L}}\hat{V}(k,x) + \hat{\pi}(k,x) = 0\}. \tag{3.7}$$

In what follow, for tractability we assume that $\beta = 1 - \alpha$ as in Abel and Eberly (1996) and change variables as $Y_t := K_t/X_t$. Then, the profit function and the value function can be rewritten as, respectively:

$$\hat{\pi}(K_t, X_t) = K_t^\alpha X_t^{1-\alpha} = Y_t^\alpha X_t = \pi(Y_t)X_t. \tag{3.8}$$

$$\hat{V}(k,x) = x\hat{V}\left(\frac{k}{x}, 1\right) = xV(y). \tag{3.9}$$

The dynamics of the variable Y is calculated as:

$$\mathrm{d}Y_t = -(\mu + m\sigma + \delta - s^2\sigma^2)Y_t\mathrm{d}t - s\sigma Y_t \mathrm{d}W_t + \mathrm{d}\zeta_t^+ - \mathrm{d}\zeta_t^-, \quad Y_{0-} = y\ (>0), \tag{3.10}$$

where $\mathrm{d}\zeta_t^{\pm} := \frac{1}{X_t}\mathrm{d}I_t^{\pm}$.

It follows from (3.9) that we have $\hat{V}_K(k,x) = V'(y)$, $\hat{V}_X(k,x) = V(y) - yV'(y)$ and $\hat{V}_{XX}(k,x) = (y^2/x)V''(y)$. The variational inequalities (3.1)–(3.4) can be also rewritten as:

$$\mathcal{L}V(y) + \pi(y) \leq 0, \tag{3.11}$$

$$V'(y) \leq p, \tag{3.12}$$

$$V'(y) \geq (1-\lambda)p, \tag{3.13}$$

$$[\mathcal{L}V(y) + \pi(y)][V'(y) - p][(1-\lambda)p - V'(y)] = 0, \tag{3.14}$$

where $\mathcal{L}$ is the operator defined by:

$$\mathcal{L} := -(\delta + \mu + m\sigma)y\frac{\mathrm{d}}{\mathrm{d}y} + \frac{1}{2}s^2\sigma^2 y^2 \frac{\mathrm{d}^2}{\mathrm{d}y^2} - (r - \mu - m\sigma). \tag{3.15}$$

The continuation region (3.7) can be rewritten as:

$$\mathcal{H} := \{y; (1-\lambda)p < V'(y) < p, \mathcal{L}V(y) + \pi(y) = 0\}. \tag{3.16}$$

The capital expansion region $\mathcal{E}$ and the capital reduction region $\mathcal{R}$ are respectively given by:

$$\mathcal{E} := \{y; (1-\lambda)p < V'(y), V'(y) = p, \mathcal{L}V(y) + \pi(y) < 0\}, \tag{3.17}$$

$$\mathcal{R} := \{y; (1-\lambda)p = V'(y), V'(y) < p, \mathcal{L}V(y) + \pi(y) < 0\}. \tag{3.18}$$

The following lemma is well-known Skorohod Lemma, which implies that Y^* is a reflected diffusion at the boundaries, $\underline{y}$ and $\bar{y}$, and a nondecreasing processes $\zeta^{\pm *}$ are the local time of Y^* at the boundaries.

Lemma 3.1. *For any $y > 0$ and given boundaries $\underline{y}$ and $\bar{y}$ with $0 < \underline{y} < \bar{y}$, there exist unique cadlag adapted process $Y^* = \{Y_t^*\}_{t \geq 0}$ and nondecreasing process $\zeta^{\pm *}$ satisfying the following Skorohod problem:*

$$\mathrm{d}Y_t^* = -(\mu + m\sigma + \delta - s^2\sigma^2)Y_t^*\mathrm{d}t - s\sigma Y_t^*\mathrm{d}W_t + \mathrm{d}\zeta_t^{+*} - \mathrm{d}\zeta_t^{-*}, \quad Y_{0-}^* = y \ (>0), \tag{3.19}$$

$$Y_t^* \in [\underline{y}, \bar{y}] \quad a.e., \quad t \geq 0, \tag{3.20}$$

$$\int_0^t \mathbf{1}_{\{Y_s^* > \underline{y}\}}\mathrm{d}\zeta_s^{+*} = 0 \ \textit{and} \ \int_0^t \mathbf{1}_{\{Y_s^* < \bar{y}\}}\mathrm{d}\zeta_s^{-*} = 0. \tag{3.21}$$

Furthermore, if $y \in [\underline{y}, \bar{y}]$, then $\zeta^{\pm *}$ is continuous. If $y < \underline{y}$ (resp. $y > \bar{y}$), then $\zeta_0^{+*} = \underline{y} - y$ (resp. $\zeta_0^{-*} = y - \bar{y}$) and $Y_0^* = \underline{y}$ (resp. $Y_0^* = \bar{y}$).

Proof. See Rogers and Williams (2000, pp. 117-118). □

The condition (3.21) means that $\zeta^{\pm *}$ increases only when Y^* reaches $\underline{y}$ or $\bar{y}$. Then, the continuation region, the capital expansion region, and the capital reduction region are replaced as follows respectively:

$$\mathcal{H} := \{y; \underline{y} < y < \bar{y}\}, \quad \mathcal{E} := \{y; y \leq \underline{y}\}, \quad \mathcal{R} := \{y; y \geq \bar{y}\}. \tag{3.22}$$

Let $\phi \in C^2$ be a function and $T < \infty$ be a stopping time. From Ito formula for cadlag semimartingales we have:

$$\begin{aligned} \mathrm{e}^{-rT} X_T \phi(Y_T) =& x\phi(y) + \int_0^T \mathrm{e}^{-rT} X_t \mathcal{L}\phi(Y_t) \mathrm{d}t + \int_0^T \mathrm{e}^{-rT} X_t [s\sigma\phi(Y_t) - s\sigma Y_t \phi'(Y_t)] \mathrm{d}W_t \\ &+ \int_0^T \mathrm{e}^{-rT} \phi'(Y_t) [\mathrm{d}I_t^{+c} - \mathrm{d}I_t^{-c}] + \sum_{0 \le t \le T} \mathrm{e}^{-rT} [\phi(Y_t) - \phi(Y_{t-})]. \end{aligned} \tag{3.23}$$

where $I_t^{\pm c} = I_t^{\pm} - \sum_{0 \le u \le t} \Delta I_s^{\pm}$ are the continuous and discontinuous parts of $I^{\pm}$.

We are now in a position to prove a solution to variational inequalities is optimal. The following theorem is well-known verification theorem. We prove the theorem by following Pham (2006, Proposition 1.3.1) and Yang and Liu (2004, Theorem 1) in Appendix A.

Theorem 3.1. *(I) Let ϕ be a solution of the variational inequalities (3.11)–(3.14) and satisfy the following:*

$$\lim_{t \to \infty} \mathrm{e}^{-rt} \phi(Y_t) = 0. \tag{3.24}$$

Then, we obtain

$$\phi(y) \ge V(y), \qquad y > 0. \tag{3.25}$$

(II) ϕ also satisfies the following:

$$\mathcal{L}\phi(y) + \pi(y) = 0, \qquad \underline{y} < y < \bar{y}, \tag{3.26}$$

$$\phi(y) = p(\underline{y} - y) + d^+, \qquad y \le \underline{y}, \tag{3.27}$$

$$\phi(y) = (1 - \lambda) p(y - \bar{y}) + d^-, \qquad y \ge \bar{y}, \tag{3.28}$$

where $d^{\pm}$ are constants. Then, there exists an optimal policy $(\zeta^{+*}, \zeta^{-*}) \in \mathcal{A}$ such that

$$\phi(y) = V(y). \tag{3.29}$$

That is, ϕ is the value function and (ζ^{+*}, ζ^{-*}) is the corresponding optimal policy.

Proof. See Appendix A. □

For $y \in \mathcal{H}$, the variational inequalities (3.11)–(3.14) lead to the following ordinary differential equation:

$$\frac{1}{2} s^2 \sigma^2 y^2 \phi''(y) - (\mu + m\sigma + \delta) y \phi'(y) - (r - \mu - m\sigma)\phi(y) + y^{\alpha} = 0. \tag{3.30}$$

The general solution of the homogeneous part of (3.30) is given by:

$$\phi(y) = A_1 y^{\gamma_1} + A_2 k^{\gamma_2}, \qquad y \in \mathcal{H}, \tag{3.31}$$

where A_1 and A_2 are constants to be determined. γ_1 and γ_2 are the solutions to the following characteristic equation:

$$\frac{1}{2} s^2 \sigma^2 \gamma^2 - \left(\mu + m\sigma + \delta + \frac{1}{2} s^2 \sigma^2 \right) \gamma - (r - \mu - m\sigma) = 0. \tag{3.32}$$

and is calculated with:

$$\begin{aligned}\gamma_1 &= \frac{1}{2} + \frac{\mu + m\sigma + \delta}{s^2\sigma^2} + \left[\left(\frac{1}{2} + \frac{\mu + m\sigma + \delta}{s^2\sigma^2}\right)^2 + \frac{2(r - \mu - m\sigma)}{s^2\sigma^2}\right]^{\frac{1}{2}} > 1,\\ \gamma_2 &= \frac{1}{2} + \frac{\mu + m\sigma + \delta}{s^2\sigma^2} - \left[\left(\frac{1}{2} + \frac{\mu + m\sigma + \delta}{s^2\sigma^2}\right)^2 + \frac{2(r - \mu - m\sigma)}{s^2\sigma^2}\right]^{\frac{1}{2}} < 0.\end{aligned} \tag{3.33}$$

On the other hand, the particular solution of (3.30) is calculated as the expected discounted value of profit function $\pi(y)$ when the firm will not invest in the capital forever:

$$\mathbb{E}\left[\int_0^\infty \mathrm{e}^{-rt}\pi(Y_t)\mathrm{d}t\right] = \frac{y^\alpha}{\rho}, \tag{3.34}$$

where $\rho := r + (\mu + m\sigma + \delta)\alpha + \frac{1}{2}s^2\sigma^2\alpha(1-\alpha)$. It follows from the assumption (2.7) that $\rho > 0$. The general solution of (3.30) is:

$$\phi(y) = A_1 y^{\gamma_1} + A_2 y^{\gamma_2} + \frac{y^\alpha}{\rho}, \qquad y \in \mathcal{H}. \tag{3.35}$$

It follows from the definition of the firm's problem that the function ϕ satisfies the following inequality:

$$\phi(y) > \frac{y^\alpha}{\rho}. \tag{3.36}$$

The first and second terms of (3.35) represent the option value to invest in capital or to reduce the capital. This implies that both constants to determine A_1 and A_2 must be positive.

Let ϕ be redefined as a candidate function of the value function and be given by:

$$\phi(y) = \begin{cases} \psi(\underline{y}) - p(\underline{y} - y), & y \le \underline{y},\\ \psi(y) := A_1 y^{\gamma_1} + A_2 y^{\gamma_2} + \frac{y^\alpha}{\rho}, & y \in (\underline{y}, \bar{y}),\\ \psi(\bar{y}) + (1-\lambda)p(y - \bar{y}), & y \ge \bar{y}. \end{cases} \tag{3.37}$$

Four unknowns A_1, A_2, $\underline{y}$ and $\bar{y}$ are determined by the following simultaneous equations:

$$\psi'(\underline{y}) = p, \tag{3.38}$$

$$\psi'(\bar{y}) = (1-\lambda)p, \tag{3.39}$$

$$\psi''(\underline{y}) = 0, \tag{3.40}$$

$$\psi''(\bar{y}) = 0. \tag{3.41}$$

Equations (3.38) and (3.39) are the smooth-pasting conditions and equations (3.40) and (3.41) are the super contact conditions (see Dumas (1991) for details).

It follows from (3.38) and (3.40) that we have the following two equations:

$$\gamma_1(\gamma_2 - \gamma_1)A_1\underline{y}^{\gamma_1 - 1} + \frac{\alpha(\gamma_2 - \alpha)\underline{y}^{\alpha-1}}{\rho} = p(\gamma_2 - 1), \tag{3.42}$$

$$\gamma_2(\gamma_1 - \gamma_2)A_2\underline{y}^{\gamma_2 - 1} + \frac{\alpha(\gamma_1 - \alpha)\underline{y}^{\alpha-1}}{\rho} = p(\gamma_1 - 1). \tag{3.43}$$

Similarly, from (3.39) and (3.41) we have the following two equations:

$$\gamma_1(\gamma_2 - \gamma_1)A_1\bar{y}^{\gamma_1 - 1} + \frac{\alpha(\gamma_2 - \alpha)\bar{y}^{\alpha-1}}{\rho} = (1-\lambda)p(\gamma_2 - 1), \tag{3.44}$$

$$\gamma_2(\gamma_1 - \gamma_2)A_2\bar{y}^{\gamma_2 - 1} + \frac{\alpha(\gamma_1 - \alpha)\bar{y}^{\alpha-1}}{\rho} = (1-\lambda)p(\gamma_1 - 1). \tag{3.45}$$

From (3.42) and (3.44), we have:

$$p(\gamma_2 - 1)\left[\left(\frac{\bar{y}}{\underline{y}}\right)^{\gamma_1} - (1-\lambda)\left(\frac{\bar{y}}{\underline{y}}\right)\right] = \frac{\alpha(\gamma_2 - \alpha)\underline{y}^{\alpha-1}}{\rho}\left[\left(\frac{\bar{y}}{\underline{y}}\right)^{\gamma_1} - \left(\frac{\bar{y}}{\underline{y}}\right)^{\alpha}\right] \tag{3.46}$$

We follow the method of Guo and Pham (2005, Section 6.3) and put $z := \frac{\bar{y}}{\underline{y}}$ (> 1). Then (3.46) can be rewritten as:

$$p(\gamma_2 - 1)\left[z^{\gamma_1} - (1-\lambda)z\right] = \frac{\alpha(\gamma_2 - \alpha)\underline{y}^{\alpha-1}}{\rho}\left[z^{\gamma_1} - z^{\alpha}\right]. \tag{3.47}$$

Similarly, from (3.43) and (3.45), we obtain that:

$$p(\gamma_1 - 1)\left[z^{\gamma_2} - (1-\lambda)z\right] = \frac{\alpha(\gamma_1 - \alpha)\underline{y}^{\alpha-1}}{\rho}\left[z^{\gamma_2} - z^{\alpha}\right]. \tag{3.48}$$

We solve the equations (3.47) and (3.48) in terms of $\frac{\alpha\underline{y}^{\alpha-1}}{\rho}$ and obtain an equation for z:

$$F(z) := \frac{\Gamma_2 z^{\gamma_1 - 1}(z^{\gamma_2} - z^{\alpha}) - \Gamma_1 z^{\gamma_2 - 1}(z^{\gamma_1} - z^{\alpha})}{\Gamma_2(z^{\gamma_2} - z^{\alpha}) - \Gamma_1(z^{\gamma_1} - z^{\alpha})} = 1 - \lambda, \tag{3.49}$$

where $\Gamma_1 := (\gamma_1 - 1)(\gamma_2 - \alpha)$ and $\Gamma_2 := (\gamma_2 - 1)(\gamma_1 - \alpha)$. It is obvious that:

$$\lim_{z\to 1} F(z) = 1, \qquad \lim_{z\to\infty} F(z) = 0. \tag{3.50}$$

Equations (3.49) and (3.50) mean that the existence of $z_\lambda > 1$ such that (3.49) holds. Therefore, the four unknowns, A_1, A_2, $\underline{y}$ and $\bar{y}$ can be expressed in the following forms:

$$A_1 = \frac{\underline{y}^{1-\gamma_1}}{\gamma_1(\gamma_2 - \gamma_1)}\left[p(\gamma_2 - 1) - \frac{\alpha(\gamma_2 - \alpha)}{\rho}\underline{y}^{\alpha-1}\right], \tag{3.51}$$

$$A_2 = \frac{\underline{y}^{1-\gamma_2}}{\gamma_2(\gamma_1 - \gamma_2)}\left[p(\gamma_1 - 1) - \frac{\alpha(\gamma_1 - \alpha)}{\rho}\underline{y}^{\alpha-1}\right], \tag{3.52}$$

$$\underline{y} = \left[\frac{\rho(\gamma_1 - 1)(z_\lambda^{\gamma_2} - (1-\lambda)z_\lambda)}{\alpha(\gamma_1 - \alpha)(z_\lambda^{\gamma_2} - z_\lambda^{\alpha})}p\right]^{\frac{1}{\alpha-1}}, \tag{3.53}$$

$$\bar{y} = z_\lambda \underline{y}. \tag{3.54}$$

We will numerically solve (3.49) and obtain the four unknowns in the next section.

4 Numerical Analysis

In this section, we numerically calculate the four unknowns, A_1, A_2, $\underline{y}$ and $\bar{y}$ and investigate the effects of changes in the parameters on the thresholds, $\underline{y}$ and $\bar{y}$. The basic parameter values are set out as follows: $r = 0.05$, $\delta = 0.1$, $\mu = 0.01$, $\sigma = 0.15$, $\alpha = 0.6$, $p = 10$, $\lambda = 0.5$, and $c = 0.4$. Then, we obtain $A_1 = 0.00014$, $A_2 = 0.49589$, $\underline{y} = 0.31654$, and $\bar{y} = 4.13044$. Figure 1 illustrates the continuation, capital investment and capital reduction regions, $\mathcal{H}$, $\mathcal{E}$ and $\mathcal{R}$ in the x-k plane, respectively.

We provide the results of the comparative static analysis of the thresholds, $\underline{y}$ and $\bar{y}$ in Figures 2–5. Figure 2 shows that the reduction region $\mathcal{R}$ is decreasing in the volatility of the demand, σ, while the continuation region $\mathcal{H}$ and the capital investment region $\mathcal{E}$ are increasing in the volatility. Overall this result implies that the incentive to wait for new information about the demand becomes stronger as the volatility increases. This is consistent with the standard result of real options analysis.

Next, Figure 3 illustrates the impact of the manager's attitude toward the ambiguity on the investment decision-making. If the firm's manager is more ambiguity averse, the ambiguity attitude parameter c is smaller. Figure 3 shows that if the firm's manager is more ambiguity averse, the capital reduction region is smaller, while the continuation region is bigger and the capital expansion region is slightly bigger. Overall, the firm's manager has an incentive to wait for new information and is more cautious in investment decision-making. Notice that when $c = 0.5$, the firm's manager is ambiguity neutral. In this case, four unknown values are: $A_1 = 0.00118$, $A_2 = 0.97722$, $\underline{y} = 0.12004$, and $\bar{y} = 1.87207$.

Next, Figure 4 depicts how the output elasticity of capital, α affects the investment decision-making. Recall that the output per capita increases in α. Figure 4 shows the continuation region is increasing in α. The reduction region is decreasing in α, while the capital investment region does not almost change.

Finally, Figure 5 explains that how the price of the capital has an impact on the investment decision-making. Figure 5 shows the price of capital has the negative impact on the continuation region and the capital investment region. On the other hand, the price has the positive effect on the capital reduction region.

These results provide useful insights into investment decision-making under the demand ambiguity.

5 Conclusion

This paper investigates a firm's partially reversible capital investment problem under the ambiguous demand. We express the ambiguous demand as the Choque-Brownian motion process. To solve the firm's problem, we formulate it as a singular stochastic control problem. Then, we use variational inequalities and derive the optimal investment strategy which is described by two thresholds to invest in and reduce the capital. Furthermore, we obtain useful insights for the firm's investment decision-making through the comparative static analysis. We find that a higher volatility and ambiguity aversion discourage the capital investment.

To conclude the paper, we suggest some possible extensions for our model. First, we could adopt another output demand like Caballero (1991), in which the firm faces an isoelastic demand function. Next, we could explore the effect of uncertainty about investment costs on the firm's investment decision-making. We also could incorporate fixed investment cost. In this case, the firm's problem is formulated by an impulse control problem. Finally, we could consider the firm's investment decision-making in duopoly market. We leave these important topics to future research.

Appendix A.

Proof of Theorem 3.1. (I) For $(\zeta^+,\zeta^-)\in\mathcal{A}$, let $T_n=\inf\{t\geq 0; Y_t\geq n\}\wedge n$, $n\in\mathbb{N}$ be the finite stopping time. We apply (3.23) between $t=0$ and $t=T_n$ and take expectation. Then, We obtain that:

$$\begin{aligned}\mathbb{E}[\mathrm{e}^{-rT_n}\phi(Y_{T_n})]=&\phi(y)+\mathbb{E}\left[\int_0^{T_n}\mathrm{e}^{-rt}\mathcal{L}\phi(Y_t)\mathrm{d}t\right]+\mathbb{E}\left[\int_0^{T_n}\mathrm{e}^{-rt}\phi'(Y_t)[\mathrm{d}\zeta_t^{+c}-\mathrm{d}\zeta_t^{-c}]\right]\\&+\mathbb{E}\left[\sum_{0\leq t\leq T_n}\mathrm{e}^{-rt}[\phi(Y_t)-\phi(Y_t^-)]\right],\end{aligned}\tag{A.1}$$

Since $p(1-\lambda)\leq\phi'(y)\leq p$ and $Y_t-Y_{t^-}=\Delta\zeta_t^+-\Delta\zeta_t^-$, the mean-value theorem implies that:

$$\phi(Y_t)-\phi(Y_{t-})=\phi'(\theta)(\Delta\zeta_t^+-\Delta\zeta_t^-)\leq p\Delta\zeta_t^+-(1-\lambda)p\Delta\zeta_t^-,\tag{A.2}$$

where $\theta\in(Y_t-Y_{t-})$. It follows from (3.11) that (A.1) is rewritten as:

$$\begin{aligned}\mathbb{E}[\mathrm{e}^{-rT_n}\phi(Y_{T_n})]\leq&\phi(y)-\mathbb{E}\left[\int_0^{T_n}\mathrm{e}^{-rt}\pi(Y_t)\mathrm{d}t\right]\\&+\mathbb{E}\left[\int_0^{T_n}\mathrm{e}^{-rt}\left[p\mathrm{d}\zeta_t^{+c}-(1-\lambda)p\mathrm{d}\zeta_t^{-c}\right]\right]\\&+\mathbb{E}\left[\sum_{0\leq t\leq T_n}\mathrm{e}^{-rt}\left[p\Delta\zeta_t^+-(1-\lambda)p\Delta\zeta_t^-\right]\right].\end{aligned}\tag{A.3}$$

Furthermore it follows from $\zeta_t^{\pm c}=\zeta_t^{\pm}-\sum_{0\leq s\leq t}\Delta\zeta_s^{\pm}$ that we have:

$$\mathbb{E}[\mathrm{e}^{-rT_n}\phi(Y_{T_n})]\leq\phi(y)-\mathbb{E}\left[\int_0^{T_n}\mathrm{e}^{-rt}\pi(Y_t)\mathrm{d}t-\int_0^{T_n}\mathrm{e}^{-rt}\left[p\mathrm{d}\zeta_t^+-(1-\lambda)p\mathrm{d}\zeta_t^-\right]\right].\tag{A.4}$$

Taking $\lim_{n\to\infty}$ and using (3.24) and the dominated convergence theorem we obtain that:

$$\phi(y)\geq\mathbb{E}\left[\int_0^{\infty}\mathrm{e}^{-rt}\pi(Y_t)\mathrm{d}t-\int_0^{\infty}\mathrm{e}^{-rt}\left[p\mathrm{d}\zeta_t^+-(1-\lambda)p\mathrm{d}\zeta_t^-\right]\right]=J(y;\zeta^+,\zeta^-).\tag{A.5}$$

From the arbitrariness of (ζ^+,ζ^-), we have:

$$\phi(y)\geq\sup_{(\zeta^+,\zeta^-)\in\mathcal{A}}J(y;\zeta^+,\zeta^-)=V(y),\tag{A.6}$$

which completes the proof of (I).

(II) For $y \in (\underline{y}, \bar{y})$, $\zeta^{\pm *}$ is continuous and increases only when $y = \underline{y}$ or $\bar{y}$. Then, for $\zeta^{\pm} = \zeta^{\pm *}$ (A.5) becomes equality:

$$\begin{aligned}\phi(y) &= \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-rt}\pi(Y_t)\mathrm{d}t - \int_0^\infty \mathrm{e}^{-rt}\left[p\mathrm{d}\zeta_t^+ - (1-\lambda)p\mathrm{d}\zeta_t^-\right]\right] \\ &= J(y; \zeta^{+*}, \zeta^{-*}) \\ &= V(y).\end{aligned} \tag{A.7}$$

For $y \leq \underline{y}$, it follows from Lemma 3.1 that we have:

$$\phi(y) = p(\underline{y} - y) + \phi(\underline{y}). \tag{A.8}$$

On the other hand, for $y \geq \bar{y}$, we have:

$$\phi(y) = (1-\lambda)p(y - \bar{y}) + \phi(\bar{y}). \tag{A.9}$$

From (A.7) we have $\phi(\underline{y}) = V(\underline{y})$ and $\phi(\bar{y}) = V(\bar{y})$. From the continuous property of $\phi(y)$, $\phi(\underline{y}) = d^+$ and $\phi(\bar{y}) = d^-$. Thus, for all $y \leq \underline{y}$ we have:

$$V(y) = p(\underline{y} - y) + \phi(\underline{y}) = \phi(y). \tag{A.10}$$

for all $y \geq \bar{y}$ we have:

$$V(y) = (1-\lambda)p(y - \bar{y}) + \phi(\bar{y}) = \phi(y). \tag{A.11}$$

This completes the proof of (II).

□

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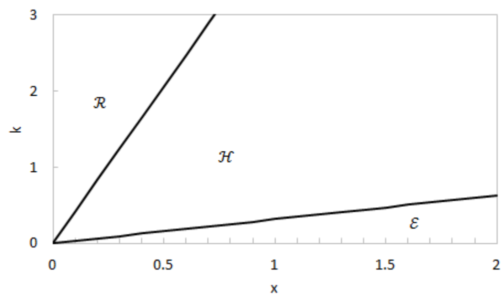

Figure 1: Continuation, expansion, and reduction regions.

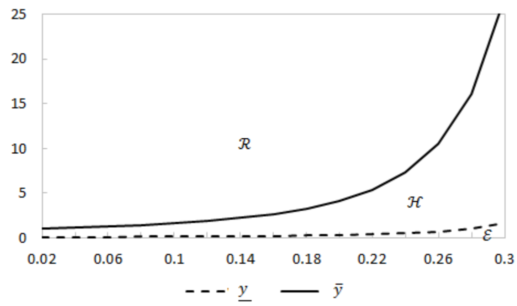

Figure 2: The effect of changing the volatility σ on the thresholds $\underline{y}$ and $\bar{y}$.

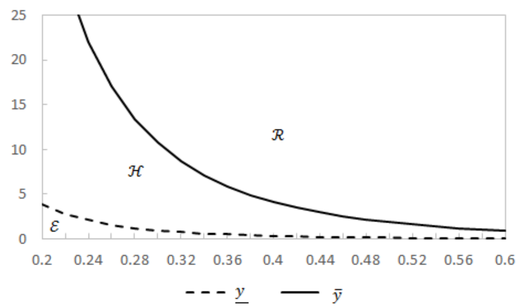

Figure 3: The effect of changing the ambiguity attitude parameter c on the thresholds $\underline{y}$ and $\bar{y}$.

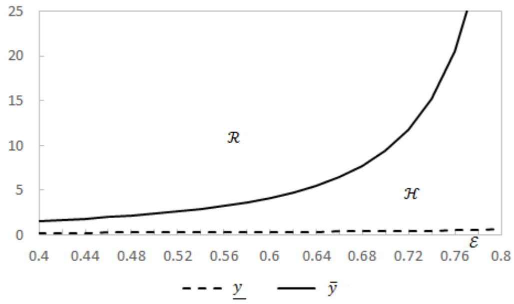

Figure 4: The effect of changing the output elasticity of capital α on the thresholds $\underline{y}$ and $\bar{y}$.

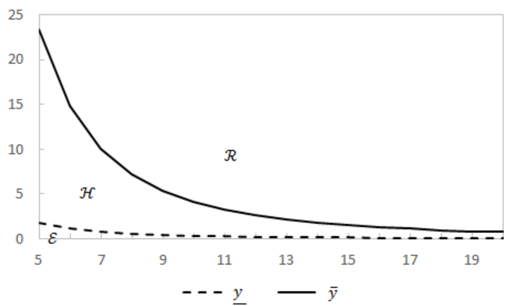

Figure 5: The effect of changing the price of capital p on the thresholds $\underline{y}$ and $\bar{y}$.