Capacity Switching Options under Rivalry and Uncertainty

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Background

- Many deregulated industries exhibit uncertainty (in prices and demand) along with competition and modularised projects
  - Examples include new technology development in mobile communications and renewable energy

- Real options approach enables a richer treatment of the decision-making problem than the now-or-never NPV one (Dixit and Pindyck, 1994)

- The impact of market characteristics on decisions and firm values can also be determined via stylised models
Related Work and Research

Objective

★ Modularity and timing
  - Dixit and Pindyck (1994), Bar-Ilan and Strange (1996), Gollier et al. (2005), Näsäkkälä and Fleten (2005), Siddiqui and Maribu (2009), Kort et al. (2010)

★ Game theory
  - Fudenberg and Tirole (1985) treat a duopoly with pre-emption over timing in a deterministic model
  - Huisman and Kort (1999) extend this to reflect market uncertainty to find that the incentive to delay in real options may be reduced due to competition
  - Possible settings: cooperative and non-cooperative (pre-emptive and non-pre-emptive)

★ Our objective is to examine the extent to which modularity can mitigate the effects of competition
Monopoly Setup

- Direct approach: obtain project of size $K_2$ for an investment cost of $I_1 + I_2$
- Sequential approach: invest in size $K_1$ before deciding to switch to a project with a higher capacity, $K_2$ (total cost is still $I_1 + I_2$)
- Market shock: $dx_t = \mu x_t dt + \sigma x_t dz_t$, where $\mu \geq 0$ and $\sigma \geq 0$
- $P_t = x_t D(\kappa_t)$ (in $/\text{unit}$), where $\kappa_t$ is the installed capacity (in units/annum) at time $t$ and $D(\kappa_t)$ is the demand parameter given the installed capacity at time $t$, where $D_1 > D_2 > D_0 = 0$
- $\rho > \mu$, $K_1 D_1 < K_2 D_2$, $\frac{I_1 + I_2}{K_2 D_2} > \frac{I_1}{K_1 D_1}$, $\frac{I_2}{K_2 D_2 - K_1 D_1} > \frac{I_1 + I_2}{K_2 D_2}$
Monopoly: Direct Approach

★ \( V^d_2(x) = \mathbb{E}_x \left[ \int_0^\infty e^{-rt} K_2 x_t D_2 dt \right] - I_1 - I_2 = \frac{xK_2D_2}{\rho-\mu} - I_2 - I_1 \)

★ Value function in state 0 obtained via \( \rho V^d_0(x) dt = \mathbb{E}_x[dV^d_0] \)

\[
\begin{align*}
\quad \frac{1}{2} \sigma^2 x^2 V^d_0''(x) + \mu x V^d_0'(x) - \rho V^d_0(x) &= 0 \\
\quad V^d_0(x) &= A^d_0 x^{\beta_1}, \text{ where } \beta_1 > 1 \text{ is the positive root to } \frac{1}{2} \xi(\xi - 1)\sigma^2 + \mu \xi - \rho = 0
\end{align*}
\]

★ Value-matching and smooth-pasting conditions:

\[
\begin{align*}
\quad V^d_0(x^d_0) &= V^d_2(x^d_0) \\
\quad \frac{dV^d_0}{dx} \bigg|_{x=x^d_0} &= \frac{dV^d_2}{dx} \bigg|_{x=x^d_0}
\end{align*}
\]

★ Solution yields \( x^d_0 = \left( \frac{\beta_1}{\beta_1 - 1} \right) \frac{(I_1 + I_2)(\rho - \mu)}{K_2D_2} \) and \( A^d_0 = \]

\[
\frac{x^d_0 - \beta_1}{\beta_1 - 1} (I_1 + I_2)
\]
Monopoly: Sequential Approach

\[ V_{1}^{s}(x) = \frac{xK_{1}D_{1}}{\rho - \mu} - I_{1} + A_{1}^{s}x^{\beta_{1}} \] if \( x < x_{1}^{s} \) and \( V_{1}^{s}(x) = V_{2}^{s}(x) \) otherwise

State-1 value-matching and smooth-pasting conditions:

- \( V_{1}^{s}(x_{1}^{s}^{-}) = V_{1}^{s}(x_{1}^{s}^{+}) \)
- \( \frac{dV_{1}^{s}}{dx} \bigg|_{x=x_{1}^{s}^{-}} = \frac{dV_{1}^{s}}{dx} \bigg|_{x=x_{1}^{s}^{+}} \)

Solution yields \( x_{1}^{s} = \left( \frac{\beta_{1}}{\beta_{1}-1} \right) \frac{I_{2}(\rho - \mu)}{[K_{2}D_{2} - K_{1}D_{1}]} > x_{0}^{d} \) and
\[ A_{1}^{s} = \frac{x_{1}^{s} - \beta_{1}I_{2}}{\beta_{1}-1} < A_{0}^{d} \]

Value function in state 0: \( V_{0}^{s}(x) = A_{0}^{s}x^{\beta_{1}} \)

- VM and SP conditions lead to \( x_{0}^{s} = \left( \frac{\beta_{1}}{\beta_{1}-1} \right) \frac{I_{1}(\rho - \mu)}{K_{1}D_{1}} < x_{0}^{d} \) and
\[ A_{0}^{s} = A_{1}^{s} + \frac{x_{0}^{s} - \beta_{1}I_{1}}{\beta_{1}-1} \]
Monopoly: Analytical Insights

★ \( x_0^d > x_0^s \)

★ \( x_1^s > x_0^d \)

★ \( x_1^s > x_0^s \)

★ \( V_0^s(x) > V_0^d(x) \quad \forall x \in (0, x_0^s) \)

★ The relative value of flexibility at \( x_0^s \), i.e.,
\[
\frac{V_0^s(x_0^s) - V_0^d(x_0^s)}{V_0^d(x_0^s)} \quad I_2^{1-\beta_1} (K_2D_2 - K_1D_1)^{\beta_1}
\]

is decreasing in \( \sigma \) as long as
\[
\left( \frac{K_2D_2 - K_1D_1}{I_2} \right) I_1^{1-\beta_1} (K_1D_1)^{\beta_1} > \left( \frac{K_2D_2}{I_1+I_2} \right) I_1^{1-\beta_1} (K_1D_1)^{\beta_1}
\]
Numerical Example: Monopoly

\[ \sigma = 0.40, \rho = 0.04, \mu = 0, I_1 = 10, I_2 = 20, K_1 = 1, K_2 = 3.5, D_{10} = 5, \]
\[ D_{11} = 4, D_{20} = 3, D_{21} = 2.5, D_{22} = 1 \]
Spillover Duopoly Setup

- Symmetric non-pre-emptive duopoly with spillover knowledge
- Direct approach: obtain project of size $K_2$ for an investment cost of $I_1 + I_2$ before follower makes similar investment
- Sequential approach: invest in size $K_1$ before waiting for follower’s entry
- Additional assumptions: $0 < D_{22} < D_{21} < D_{20} \equiv D_2 < D_{11} < D_{10} \equiv D_1$, $K_2D_{22} > K_1D_{21}$, $K_2D_{21} > K_1D_{11}$, and $\frac{1}{2}(K_1 + K_2)D_{21} > K_1D_{11}$
Spillover Duopoly: Direct Approach

★ Value functions:

- \( V_{22}^{j,d}(x) = \frac{xK_2D_{22}}{\rho - \mu} - I_2 - I_1 \)
- \( V_{20}^{L,d}(x) = \frac{xK_2D_{20}}{\rho - \mu} - I_2 - I_1 + A_{20}^{L,d}x^{\beta_1} \)
- \( V_{20}^{F,d}(x) = A_{20}^{F,d}x^{\beta_1} \)
- \( V_{00}^{j,d}(x) = A_{00}^{j,d}x^{\beta_1} \)

★ VM and SP conditions:

- \( V_{20}^{F,d}(x_{20}^{d}) = V_{22}^{F,d}(x_{20}^{d}) \)
- \( \left. \frac{dV_{20}^{F,d}}{dx} \right|_{x=x_{20}^{d}} = \left. \frac{dV_{22}^{F,d}}{dx} \right|_{x=x_{20}^{d}} \)
- \( V_{20}^{L,d}(x_{20}^{d}) = V_{22}^{L,d}(x_{20}^{d}) \)
- \( V_{00}^{j,d}(x_{00}^{d}) = \frac{1}{2} \left[ V_{20}^{L,d}(x_{00}^{d}) + V_{20}^{F,d}(x_{00}^{d}) \right] \)
- \( \left. \frac{dV_{00}^{j,d}}{dx} \right|_{x=x_{00}^{d}} = \frac{1}{2} \left[ \left. \frac{dV_{20}^{L,d}}{dx} \right|_{x=x_{00}^{d}} + \left. \frac{dV_{20}^{F,d}}{dx} \right|_{x=x_{00}^{d}} \right] \)
Spillover Duopoly: Direct Approach Solutions

\[ x_{20}^d = \left( \frac{\beta_1}{\beta_1 - 1} \right) \frac{(I_1 + I_2)(\rho - \mu)}{K_2 D_{22}} \]

\[ A_{20}^{F,d} = \frac{x_{20}^d - \beta_1 (I_1 + I_2)}{\beta_1 - 1} \]

\[ A_{20}^{L,d} = \frac{x_{20}^d - \beta_1 (I_1 + I_2)(D_{22} - D_{20}) \beta_1}{(\beta_1 - 1) D_{22}} \]

\[ x_{00}^d = \left( \frac{\beta_1}{\beta_1 - 1} \right) \frac{(I_1 + I_2)(\rho - \mu)}{K_2 D_{20}} = x_0^d \]

\[ A_{00}^{j,d} = \frac{1}{2} \left[ A_{20}^{L,d} + A_{20}^{F,d} + \frac{x_{00}^d - \beta_1 (I_1 + I_2)}{\beta_1 - 1} \right] \]
Spillover Duopoly: Sequential Approach

- Value functions:
  \[ V_{22}^j,s(x) = \frac{xK_2D_{22}}{\rho-\mu} - I_2 - I_1, \]
  \[ V_{21}^L,s(x) = \frac{xK_2D_{21}}{\rho-\mu} - I_1 - I_2 + A_{21}^{L,s}x^{\beta_1}, \]
  \[ V_{21}^F,s(x) = \frac{xK_1D_{21}}{\rho-\mu} - I_1 + A_{21}^{F,s}x^{\beta_1}, \]
  \[ V_{11}^L,s(x) = \frac{xK_1D_{10}}{\rho-\mu} - I_1 + A_{10}^{L,s}x^{\beta_1}, \]
  \[ V_{10}^F,s(x) = A_{10}^{F,s}x^{\beta_1}, \]
  \[ V_{00}^j,s(x) = A_{00}^{j,s}x^{\beta_1} \]

- Some VM and SP conditions:
  - \[ V_{21}^{F,s}(x_{21}) = V_{22}^{F,s}(x_{21}) \]
  - \[ \frac{dV_{21}^{F,s}}{dx}|_{x=x_{21}^s} = \frac{dV_{22}^{F,s}}{dx}|_{x=x_{21}^s} \]
  - \[ V_{21}^{L,d}(x_{21}) = V_{22}^{L,s}(x_{21}) \]
  - \[ V_{11}^{j,s}(x_{11}^s) = \frac{1}{2} \left[ V_{21}^{L,s}(x_{11}^s) + V_{21}^{F,s}(x_{11}^s) \right] \]
  - \[ \frac{dV_{11}^{j,s}}{dx}|_{x=x_{11}^s} = \frac{1}{2} \left[ \frac{dV_{21}^{L,s}}{dx}|_{x=x_{11}^s} + \frac{dV_{21}^{F,s}}{dx}|_{x=x_{11}^s} \right] \]
Spillover Duopoly: Sequential Approach Solutions

\[
x_{21}^s = \frac{\beta_1}{\beta_1 - 1} \frac{I_2(\rho - \mu)}{K_2 D_{22} - K_1 D_{21}}
\]

\[
A_{21}^{F,s} = \frac{x_{21}^s - \beta_1 I_2}{\beta_1 - 1}, \quad A_{21}^{L,s} = \frac{x_{21}^s - \beta_1 I_2}{\beta_1 - 1} \left[ \frac{K_2 D_{22} - K_2 D_{21}}{K_2 D_{22} - K_1 D_{21}} \right]
\]

\[
x_{11}^s = \frac{\beta_1}{\beta_1 - 1} \frac{I_2(\rho - \mu)}{2K_1 D_{11} - (K_1 + K_2)D_{21}}
\]

\[
A_{11}^{j,s} = \frac{1}{2} \left( A_{21}^{L,s} + A_{21}^{F,s} + \frac{(x_{11}^s)^{-\beta_1 I_2}}{\beta_1 - 1} \right)
\]

\[
x_{10}^s = \frac{\beta_1}{\beta_1 - 1} \frac{I_1(\rho - \mu)}{K_1 D_{11}}
\]

\[
A_{10}^{F,s} = A_{11}^{j,s} + \frac{x_{10}^s - \beta_1 I_1}{\beta_1 - 1}, \quad A_{10}^{L,s} = A_{11}^{j,s} + \frac{x_{10}^s - \beta_1 I_1}{\beta_1 - 1} \frac{D_{11} - D_{10}}{D_{11}(\beta_1 - 1)}
\]

\[
x_{00}^s = \frac{\beta_1}{\beta_1 - 1} \frac{I_1(\rho - \mu)}{K_1 D_{10}} = x_0^s
\]

\[
A_{00}^{j,s} = \frac{1}{2} \left( A_{10}^{L,s} + A_{10}^{F,s} + \frac{x_{00}^s - \beta_1 I_1}{\beta_1 - 1} \right)
\]
Spillover Duopoly: Analytical Insights

★ There exists a region where $V_{20}^{L,d}(x)$ decreases in $x$ when
\[ \frac{D_{20}-D_{22}}{D_{20}} > \frac{1}{\beta_1} \]

- Note that $V_{20}^{L,d}(x) = \frac{xK_2D_{20}}{\rho - \mu} - I_1 - I_2 + \left( \frac{x}{x_{20}} \right)^{\beta_1} x_{20}^d \left[ \frac{K_2D_{22} - K_2D_{20}}{\rho - \mu} \right]$

★ Value of duopolist is less than that of a monopolist

★ The relative value of a duopolist firm to the monopolist at $x_0^s$, i.e., $\frac{V_{j0}^{d}(x_0^s)}{V_0^d(x_0^s)}$, is decreasing in $\sigma$ as long as
\[ \left( \frac{D_{20}}{D_{22}} \right)^{\beta_1} - \frac{D_{22}}{D_{20}-D_{22}} > \epsilon \]

- Monopolist’s value increases in $\sigma$ due to greater value of waiting
- Duopolist’s value:
  - Increases in $\sigma$ for the same reason
  - Increases in $\sigma$ due to postponed entry of rival
  - Decreases in $\sigma$ due to greater impact of rival’s entry

Value of firm
Numerical Example: Spillover Duopoly

Duopoly direct value curves for $\sigma = 0.40$

Duopoly sequential value curves for $\sigma = 0.40$
Numerical Example: Spillover Duopoly with Low Loss in Market Share
Numerical Example: Spillover Duopoly Thresholds

![Graph showing the relationship between market shock and volatility for different threshold values.](image)
Numerical Example: Spillover
Duopoly Value of Flexibility

\[ \frac{V^{s}_{0}(x^{s}_{0}) - V^{d}_{0}(x^{s}_{0})}{V^{d}_{0}(x^{s}_{0})} \]
Numerical Example: Spillover Duopoly Effect of Competition

\[
\frac{V_{00}^{j,d}(x_0^s)}{V_0^d(x_0^s)} \quad \text{or} \quad \frac{V_{00}^{j,s}(x_0^s)}{V_0^s(x_0^s)}
\]
Numerical Example: Spillover Duopoly Effect of Competition with Low Loss in Market Share

\[ \frac{V_{00}^{j,d}(x_0^s)}{V_0^d(x_0^s)} \text{ or } \frac{V_{00}^{j,s}(x_0^s)}{V_0^s(x_0^s)} \]
Proprietary Duopoly Setup
Proprietary Duopoly: Sequential Approach

★ Value functions: 
\[ V_{22}^j(s) (x) = \frac{xK_2D_{22}}{\rho - \mu} - I_2 - I_1, \]
\[ V_{21}^L(s) (x) = \frac{xK_2D_{21}}{\rho - \mu} - I_1 - I_2 + A_{L,s}^L x_{\beta_1}, \]
\[ V_{21}^F(s) (x) = \frac{xK_1D_{21}}{\rho - \mu} - I_1 + A_{F,s}^L x_{\beta_1}, \]
\[ V_{20}^L(s) (x) = \frac{xK_2D_{20}}{\rho - \mu} - I_1 - I_2 + A_{L,s}^L x_{\beta_1}, \]
\[ V_{20}^F(s) (x) = A_{20}^F x_{\beta_1}, \]
\[ V_{10}^L(s) (x) = \frac{xK_1D_{10}}{\rho - \mu} - I_1 + A_{L,s}^L x_{\beta_1}, \]
\[ V_{11}^F(s) (x) = A_{00}^F x_{\beta_1} \]

★ Some VM and SP conditions:

- \[ V_{20}^F(s) (x_{20}) = V_{21}^F(s) (x_{20}) \]
- \[ \frac{dV_{20}^F(s)}{dx} |_{x=x_{20}} = \frac{dV_{21}^F(s)}{dx} |_{x=x_{20}} \]
- \[ V_{20}^L(s) (x_{20}) = V_{21}^L(s) (x_{20}) \]
- \[ V_{10}^L(s) (x_{10}) = V_{20}^L(s) (x_{10}) \]
- \[ \frac{dV_{10}^L(s)}{dx} |_{x=x_{10}} = \frac{dV_{20}^L(s)}{dx} |_{x=x_{10}} \]
Numerical Example: Proprietary Duopoly

Proprietary-knowledge duopoly sequential value curves for $\sigma = 0.40$
Numerical Example: Proprietary Duopoly Value of Flexibility

\[
\frac{V_s^0(x_s^0) - V_d^0(x_s^0)}{V_d^0(x_s^0)}
\]
Numerical Example: Proprietary Duopoly Effect of Competition

\[
\frac{V_{00}^{j,d}(x_0^s)}{V_0^d(x_0^s)} \quad \text{or} \quad \frac{V_{00}^{j,s}(x_0^s)}{V_0^s(x_0^s)}
\]
Summary

★ Model a two-stage investment game under uncertainty and with discretion over timing
  ▶ Captures aspects of industries with new technology development: uncertainty, lumpy and modularised investment, and competition

★ Use a dynamic programming approach with game theory in a symmetric, non-pre-emptive framework
  ▶ Relative value of flexibility generally worth more for a duopoly and decreases with uncertainty
  ▶ Effect of competition is relatively higher under a direct approach and more severe with uncertainty for a high loss in market share
  ▶ Proprietary duopoly setup reduces the value of a typical firm

★ Future work: pre-emptive duopoly, asymmetric firms, and uncertain technology arrival
Questions